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SKEW IMPACT OF A MATERIAL POINT ON AN INFINITE STRING LYING ON AN ELASTIC SUPPORT*

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The skew impact of a material point on an infinite string resting on a linearly elastic support is considered. An exact solution of the problem is obtained for the case when the string is not spring-loaded. In the case of a spring-loaded string the paper deals with the functional relations between the duration of contact, the angle of reflection, the distance covered by a material point along the string during the contact, and the coefficient of restitution and the angle of impact of the point on the string, for various degrees of stiffness of the support.

In problems of the collision between a concentrated object and a one-dimensional system discussed in /1-4/ mainly direct impact was investigated. The present paper is concerned with the case when the concentrated object slides over a distributed system.

Let us consider the skew impact of a material point of mass m on an infinite string at rest, of density ρ and tension T, lying on an elastic support of stiffness k. We shall assume that the motions of the object take place in the xu plane in such a manner that $u(x,t)$ is the transverse deflection of the string, and $u_0(t)$ and $l(t)$ are the generalized coordinates of the point along the u and x axes, respectively, i.e. the relation $u_0(t)=u(l(t),t)$ holds at the time of contact. We can assume without loss of generality that the contact begins at the instant $t=0$ at the point $x=0$.

Assuming that the oscillations of the string are small, we can formulate the problem of their simultaneous motion, as in /2/, as follows:

$$u_{tt}-a^2u_{xx}+h^2u=0 \quad (a^2=T/\rho,\ h^2=k/\rho) \tag{1}$$

$$mu_0^{\cdot\cdot}(t)=\rho(a^2-l^{\cdot 2}(t))[u_x]$$

$$ml^{\cdot\cdot}(t)=-{}^1/_2\rho(a^2-l^{\cdot 2}(t))[u_x^2] \tag{2}$$

$$([A(x,t)]=A(l(t)+0,t)-A(l(t)-0,t)),$$

$$u(x,0)=0,\ u_t(x,0)=0 \tag{3}$$

$$u_0(0)=0,\ u_0^{\cdot}(0)=-v_1,\ l(0)=0,\ l^{\cdot}(0)=v_2\ (|v_2|<a)$$

$$u(x,t)=\begin{cases} u_-(x,t), & -\infty<x<l(t)\\ u_0(t), & x=l(t)\\ u_+(x,t), & l(t)<x<+\infty\end{cases}$$

where $u(x,t)$ is a piecewise-smooth function.

To solve the problem we shall use the following integral Fourier transformation in x:

$$U(\omega,t)=\int\limits_{-\infty}^{+\infty}u(x,t)e^{-2\pi i\omega x}dx=F(u(x,t))$$

Taking into account the relations

$$F(u_{tt})=U_{tt}-l^{\cdot 2}(t)V,\ F(u_{xx})=-4\pi^2\omega^2U-V$$

$$V=m\rho^{-1}u_0^{\cdot\cdot}(t)(a^2-l^{\cdot 2}(t))^{-1}e^{-2\pi i\omega l(t)}$$

we shall transform the first equation of (1) into the ordinary differential equation

$$U_{tt}+(4\pi^2\omega^2a^2+h^2)U=-m\rho^{-1}u_0^{\cdot\cdot}(t)e^{-2\pi i\omega l(t)}$$

with zero initial conditions. Solving this equation and returning to the original function, we obtain /5/

$$u(x,t)=-\frac{m}{2\rho a}\int\limits_0^t u_0^{\cdot\cdot}(\tau)J_0\left(h\sqrt{(t-\tau)^2-\frac{(x-l(\tau))^2}{a^2}}\right)\times \tag{4}$$

$$\theta(a(t-\tau)-|x-l(\tau)|)d\tau,\quad |x|<at$$

$$u(x,t)=0,\ |x|>at$$

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Here $J_0(y)$ is a Bessel function and $\theta(y)$ is the Heaviside unit function. Since $u(x, t) \equiv 0$ when $|x| > at$, we shall continue our investigation in the region $|x| < at$.

We rewrite Eq.(2) in the form

$$\frac{a + l^{\cdot}(t)}{a - l^{\cdot}(t)} = \frac{a + l^{\cdot}(0)}{a - l^{\cdot}(0)} \exp\left(-\frac{\rho a}{m}\int_0^t [u_x^2]\, dt\right)$$

From this it follows that if $|l^{\cdot}(0)| < a$, the following inequality holds for the function $l(t)$ satisfying Eq.(2):

$$|l^{\cdot}(t)| \leqslant a \tag{5}$$

We can show that for $\tau \in [0, t]$

$$\theta(a(t-\tau) - |x - l(\tau)|) = \begin{cases} \theta(a(t-\tau) - x + l(\tau)), & x > l(t) \\ \theta(a(t-\tau) + x - l(\tau)), & x < l(t) \end{cases} \tag{6}$$

Indeed, if $x > l(t)$ and $x > l(\tau)$, then Eq.(6) is obvious. If $l(t) < x < l(\tau)$, then using inequality (5) we arrive at the following sequence of inequalities:

$$0 < l(\tau) - x < l(\tau) - l(t) \leqslant |l^{\cdot}(\xi)|\,(t-\tau) \leqslant a(t-\tau) \quad (\tau \leqslant \xi \leqslant t)$$

From this we have $a(t-\tau) \pm (x - l(\tau)) > 0$, i.e.

$$\theta(a(t-\tau) - |x - l(\tau)|) = \theta(a(t-\tau) - x + l(\tau))$$

Relation (6) is proved for $x < l(t)$ in the same manner.

From relations (4) and (6) we obtain

$$u_{\pm}(x, t) = -\frac{m}{2\rho a}\int_0^{\tau_{\pm}} u_0^{\cdot\cdot}(\tau) J_0\left(h\left[(t-\tau)^2 - \frac{(x - l(\tau))^2}{a^2}\right]^{1/2}\right) d\tau \tag{7}$$

where $\tau_{\pm}$ are obtained from the relations $t - \tau_{\pm} = \pm(x - l(\tau_{\pm}))/a$.

We note that $\tau_- = \tau_+ = t$ when $x = l(t)$.

Using the condition of continuity $u_0(t) = u(l(t), t)$ and boundary condition (2), we obtain from (7) the following non-linear system of integrodifferential equations:

$$u_0(t) = -\frac{m}{2\rho a}\int_0^t u_0^{\cdot\cdot}(\tau) J_0(hy)\, d\tau \tag{8}$$

$$l^{\cdot\cdot}(t) = -\frac{m}{2\rho a} u_0^{\cdot\cdot}(t)\left(\frac{u_0^{\cdot\cdot}(t)\, l^{\cdot}(t)}{a^2 - l^{\cdot 2}(t)} - \frac{h}{a^2}\int_0^t u_0^{\cdot\cdot}(\tau)\frac{J_1(hy)}{y}(l(t) - l(\tau))\, d\tau\right)$$

$$(y = [(t-\tau)^2 - (l(t) - l(\tau))^2/a^2]^{1/2})$$

with initial conditions (3).

Let us consider in more detail the limiting case when $h = 0$, corresponding to the skew impact of a material point on a string without elastic support. The system of Eqs.(8) will now be rewritten in the form

$$u_0(t) = -\frac{m}{2\rho a}\int_0^t u_0^{\cdot\cdot}(\tau)\, d\tau, \quad l^{\cdot\cdot}(t) = -\frac{m \ddot{u}_0^2(t)\, l^{\cdot}(t)}{2\rho a(a^2 - l^{\cdot 2}(t))}$$

Integrating this system we arrive at the following relations:

$$u_0(t) = \frac{v_1 m}{2\rho a}\left[\exp\left(-\frac{2\rho a t}{m}\right) - 1\right]$$

$$a^2 \ln\frac{l^{\cdot}(t)}{v_2} - \frac{l^{\cdot 2}(t) - v_2^2}{2} = -\frac{v_1^2}{2}\left[1 - \exp\left(-\frac{4\rho a t}{m}\right)\right], \quad v_2 \neq 0$$

$$l(t) \equiv 0, \; v_2 = 0$$

The time of contact t^* is found from the relation /2/ $u_0^{\cdot\cdot}(t^*) = 0$, and in the limiting case in question the time is infinite. The velocity of the material point $l^{\cdot}(t)$ tends asymptotically to the quantity v^* given by the relation

$$\ln(v^*/v_2) = (v^{*2} - v_1^2 - v_2^2)/2a^2 \; (v_2 \neq 0)$$

When $h \neq 0$, the analytic investigation of system (8) becomes considerably more difficult; we therefore solved it by numerical methods. In order to facilitate the interpretation of the numerical results, we shall introduce the following notation:

$$\varphi_0 = \operatorname{arctg}\frac{v_2}{v_1}, \quad \varphi^* = \operatorname{arctg}\frac{l^{\cdot}(t^*)}{u_0^{\cdot}(t^*)}, \quad \tau^* = \frac{2\rho a t^*}{m}$$

$$z^* = \frac{2\rho l(t^*)}{m}, \quad K = \frac{l^{\cdot 2}(t^*) + u_0^{\cdot 2}(t^*)}{v_1^2 + v_2^2}, \quad \beta = \frac{hm}{2\rho a}$$

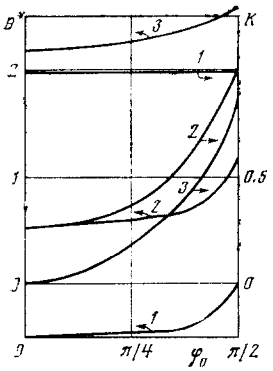

Fig.1

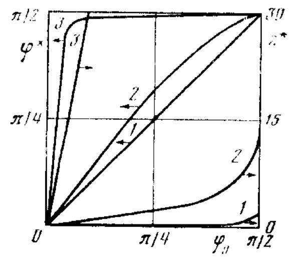

Fig.2

Here φ_0 is the angle of impact of the material point on the string, φ^* is the angle of reflection, τ^* is the dimensionless time of contact, z^* is the dimensionless path of the material point along the string during the time of contact, K is the energy coefficient of restitution and β is the dimensionless stiffness coefficient of the support.

Figs.1 and 2 show the relations $\theta^* = \lg\tau^*(\varphi_0)$, $K = K(\varphi_0)$, $\varphi^* = \varphi^*(\varphi_0)$, $z^* = z^*(\varphi_0)$ at $\beta = 100, 1, 0.01$ (curves *1*, *2* and *3* respectively). The relations given above were obtained for $V_0 = \sqrt{v_1^2 + v_2^2} = a$. The qualitative form of the graphs remains unchanged at other values of the initial velocity $V_0 < a$

Analysing the results obtained we arrive at the following conclusions:

a) an increase in the angle of impact leads to an increase in the time of contact, the distance covered by the material point, and the coefficient of restitution;

b) the inquality $\varphi^* \geqslant \varphi_0$ holds for any $\beta \neq 0$;

c) when β decreases, we have $\tau^* \to +\infty, z^* \to \infty$

$$K \to 0,\ \varphi^* \to \pi/2;$$

d) when $\beta \to +\infty$, we have $\tau^* \to 0, z^* \to 0, K \to 1, \varphi^* \to \varphi_0$.

As might have been expected, the collision of a material point with a string which is not spring-loaded ($h = 0$) is absolutely non-elastic, and numerical investigation of the impact on a spring-loaded string shows that, as the rigidity of the support increases, the character of the collision approaches absolutely elastic.

REFERENCES

1. RAKHMATULLIN KH.A., On the skew impact against a flexible string at high velocities with friction. PMM, 9, 6, 1945.
2. MALANOV S.B. and UTKIN G.A., Shock interaction between a concentrated object and a one-dimensional elastic system. PMM, 52, 1, 1988.
3. BIDERMAN V.L., Theory of Mechanical Oscillations. Vyssh. Shk. Moscow, 1980.
4. LUR'E A.I., Operational Calculus and its Application in Problems of Mechanics. Gostekhizdat, Moscow, Leningrad, 1951.
5. PRUDNIKOV A.P., BRYCHKOV YU.A. and MARICHEV O.I., Integrals and Series. Elementary Functions. Nauka, Moscow, 1981.

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